Summer School
Shanghai Observatory

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Outline

- Linear expansion
  - Vector space
  - Fourier analysis
  - Spherical harmonics
  - EOF/PCA
- Normal modes
  - of musical instruments
  - of Earth
- Inverse problems
- Earth's rotation
  - "Astronomical"
  - "Geophysical"
- Gravity and Geomagnetism

Vector space

- Dimensionality
- Addition
- Null vector
- Scaling / multiplication
- Unit vector
- Inner product
- norm
- Basis
- Projection / component

Tensor of degree *n*

- Scalar (*n* = 0)
- Vector (*n* = 1)
- Tensor of *n* = 2; (matrix)
  – stress
  – strain
- Tensor of *n* = 4:
  – elasticity / compliance

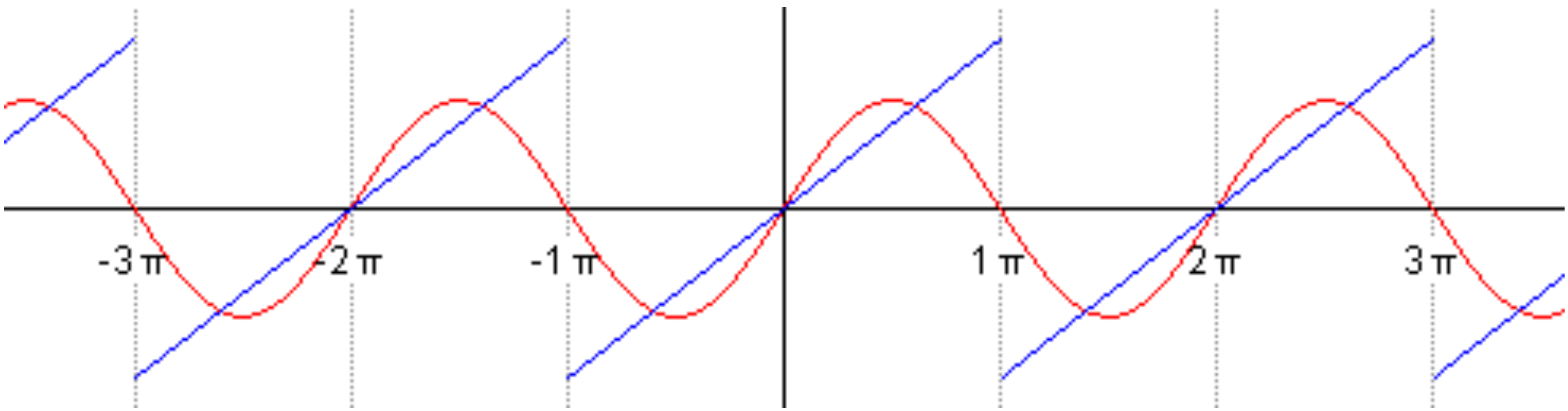
$$\sigma_{ij} = c_{ijk\ell} \; \epsilon_{k\ell}$$

Hilbert space

- “Function space”
- Infinite dimension
- domain
- Inner product
- Orthogonality
- Basis function
- Completeness

Fourier analysis

- Basis function = sinusoids
- Cartesian coordinates of dimension *n*
- Orthogonal
- Complete


-3 π
-2 π
-1 π
1 π
2 π
3 π

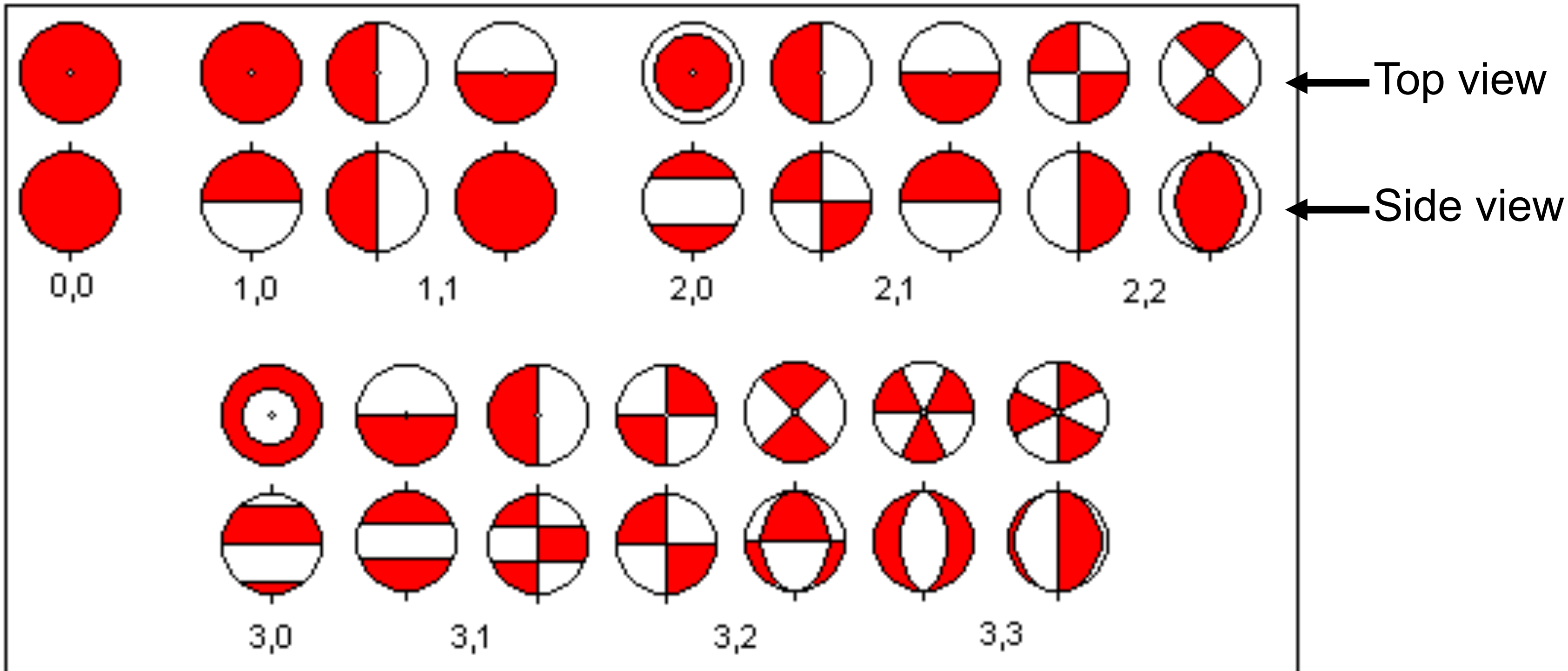
Spherical Harmonics

- Spherical coordinates
- Satisfying Laplace equation
- Solid harmonics (3-D)
- Surface harmonics (2-D)
- Legendre functions
- Orthogonal
- Complete

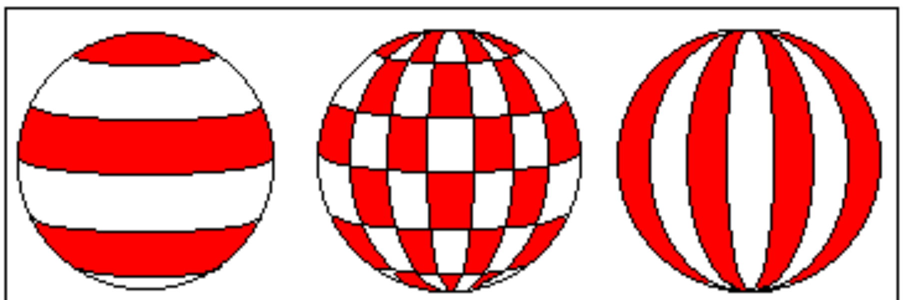
$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

$$\begin{cases} Y_{00} = \dfrac{1}{(4\pi)^{\frac{1}{2}}} \\ Y_{10} = \left(\dfrac{3}{4\pi}\right)^{\frac{1}{2}} \cos(\theta) \\ Y_{1\pm1} = \mp\left(\dfrac{3}{8\pi}\right)^{\frac{1}{2}} \sin(\theta)\, e^{\pm i\phi} \\ Y_{20} = \left(\dfrac{5}{16\pi}\right)^{\frac{1}{2}} (3\cos^2(\theta)-1) \\ Y_{2\pm1} = \mp\left(\dfrac{15}{8\pi}\right)^{\frac{1}{2}} \cos(\theta)\sin(\theta)\, e^{\pm i\phi} \\ Y_{2\pm2} = \left(\dfrac{15}{32\pi}\right)^{\frac{1}{2}} \sin^2(\theta)\, e^{\pm 2i\phi} \end{cases}$$

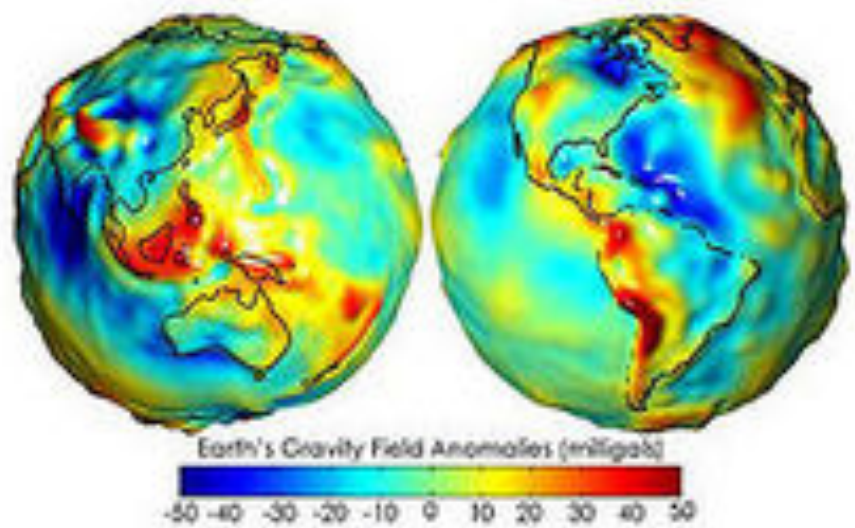
$$\int_0^{2\pi}\int_0^{\pi} (Y_l^m)^* Y_{l'}^{m'} \sin\theta d\theta\, d\phi = \delta_{ll'}\delta_{mm'}$$



Spherical Harmonics, Nodal Surfaces n = 0,1,2,3



Types of Spherical Harmonics

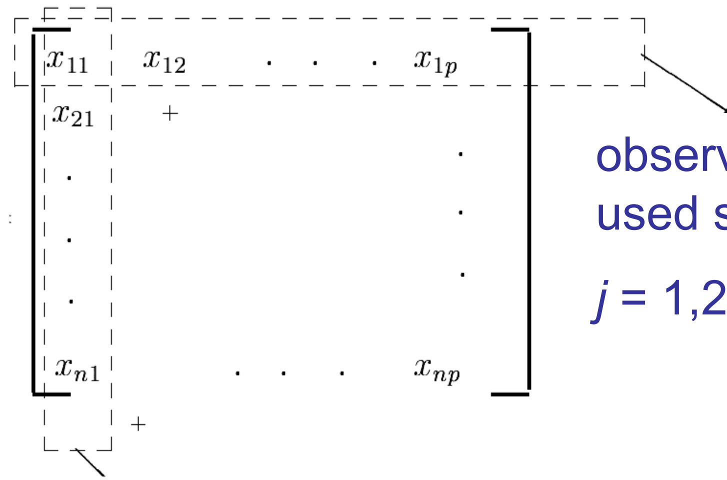


EOF (Empirical Orthogonal Function)

- decomposing data matrix into "mode of standing-oscillations"

- presented by spatial pattern and temporal series.

Data matrix $\boldsymbol{D}(\boldsymbol{x},t)$:



observations from all the used stations at time *j*.

$j = 1,2,\ldots n$

time series for station x_i, $i = 1,2,\ldots p$

Decomposed: $\boldsymbol{D}(x, t) = \Sigma_i\, S_i(x)\, T_i(t)$,

EOF = the eigen-solutions of the covariance matrix of D : $R = D^T D$

Spatial pattern $S_i(x)$ **= eigenvectors (orthogonal)**

Time series $T_i(t)$ **= projection of *F* onto the i-th eigenvector (orthogonal)**

% variance **= eigen-value**

Normal modes

- Musical instrument
  - Wave equation
  - Boundary condition
  - Propagating wave / normal mode duality
  - 1-D: string
  - 2-D: drum
  - 3-D:
- Earth

What is music?

Satisfying the wave equation (*n*-D) under boundary conditions:

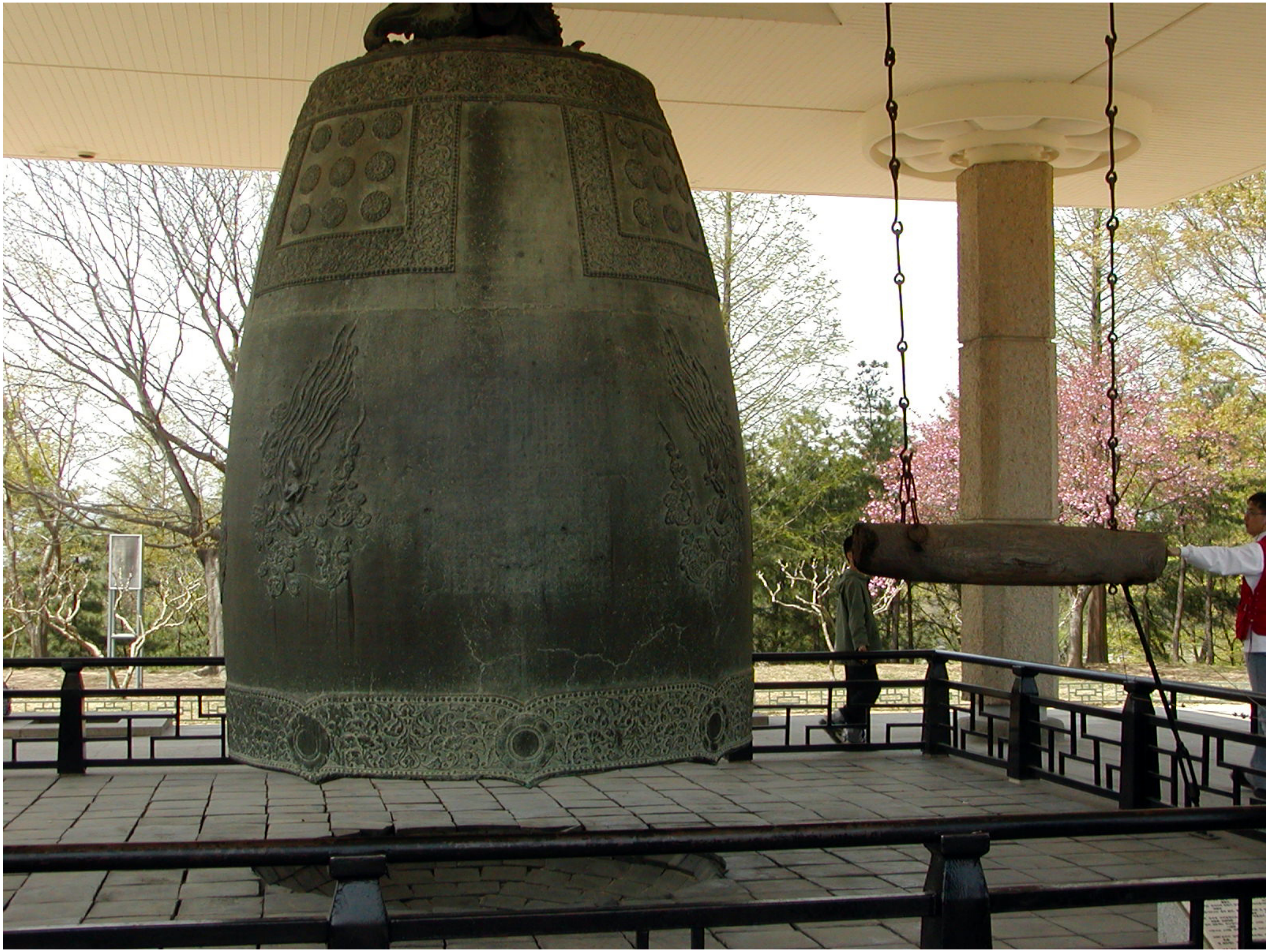
Oscillation of the bell (**r**,*t*)

= **Σ** (of all normal modes, [illegible])

amplitude (depending on where, how hard you strike, etc., called "excitation".)

* normal-mode eigenfunction (**r**) (depending on the physical property of the bell, e.g, if symmetric, sinusoids in 1-D, Legendre or Bessel functions in 2-D, etc. Earth is 3-D = 2-D + 1-D.)

**exp*(*i*ω*t*) (ω is the normal-mode eigenfrequency, or "natural" resonance frequency = music tones, with imaginary part = natural decay. Quantized because of boundary conditions.)



A typical seismogram

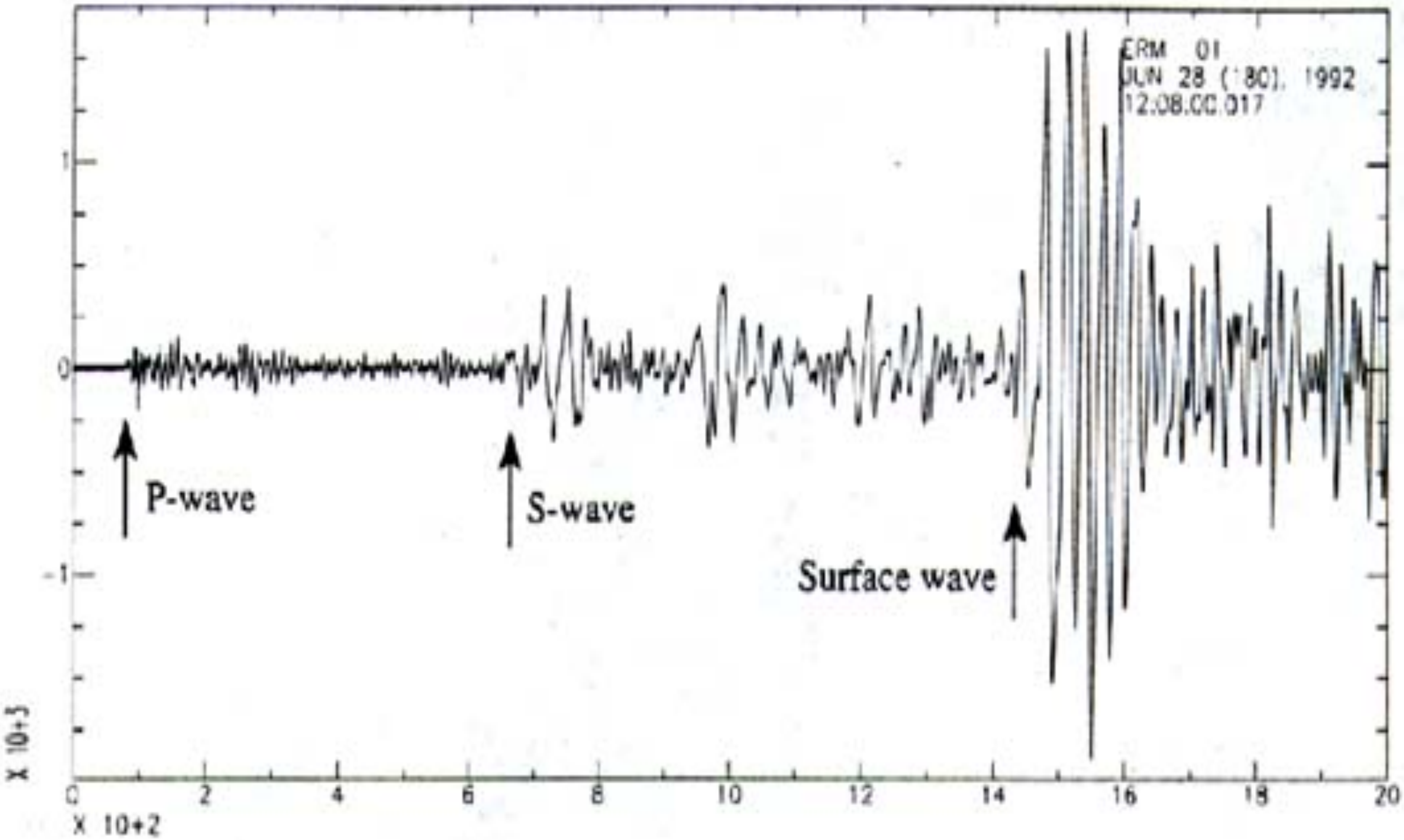
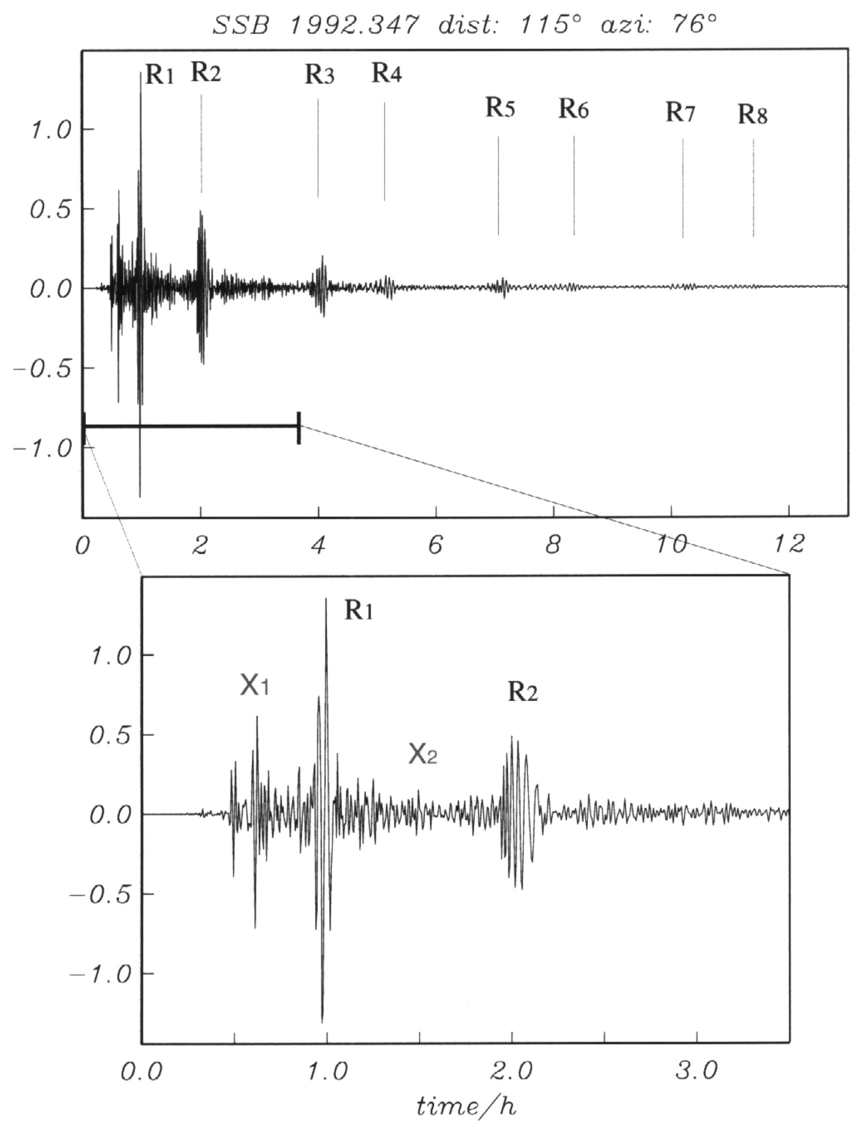
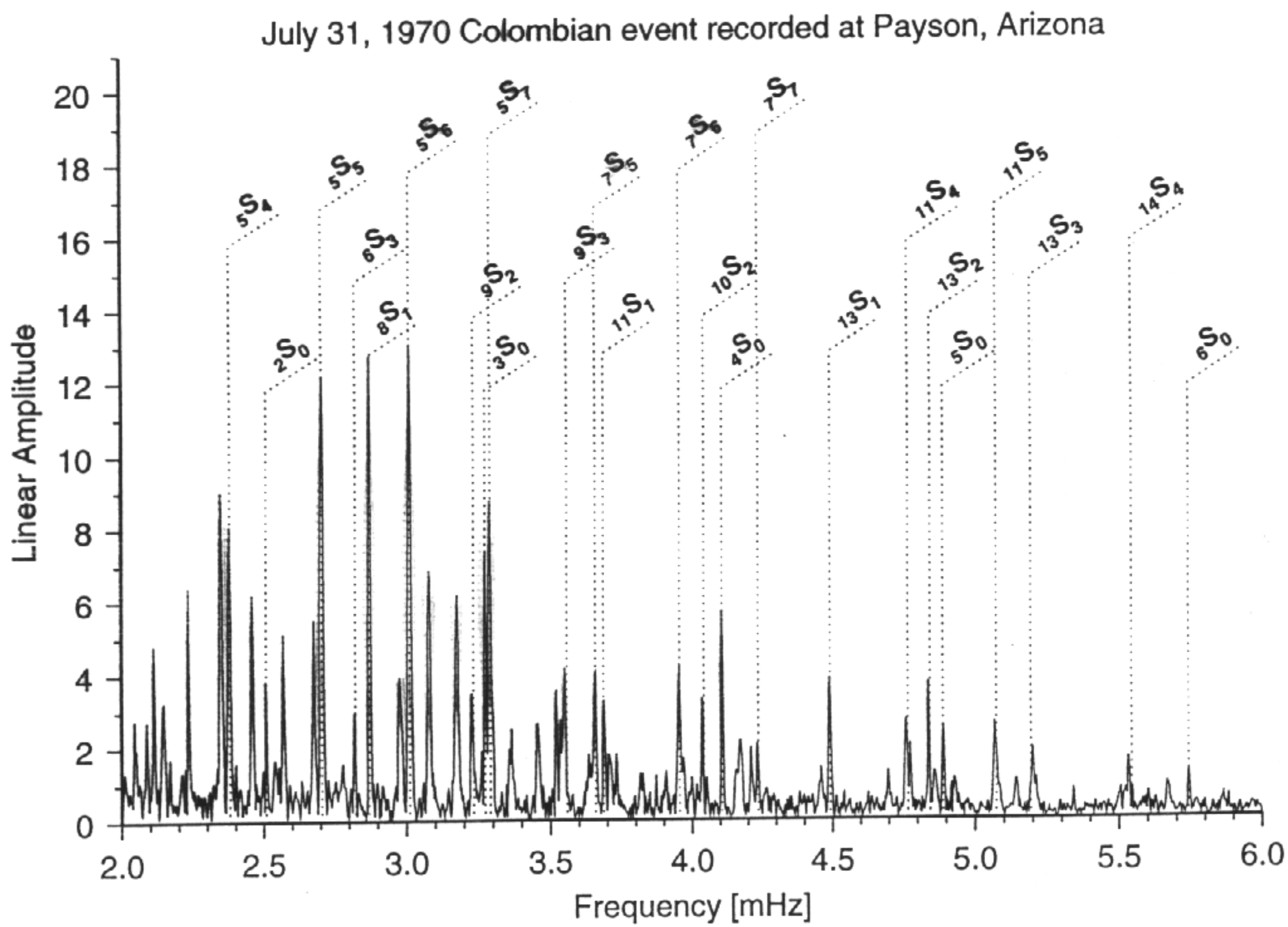


FIGURE 6.4 A seismogram.

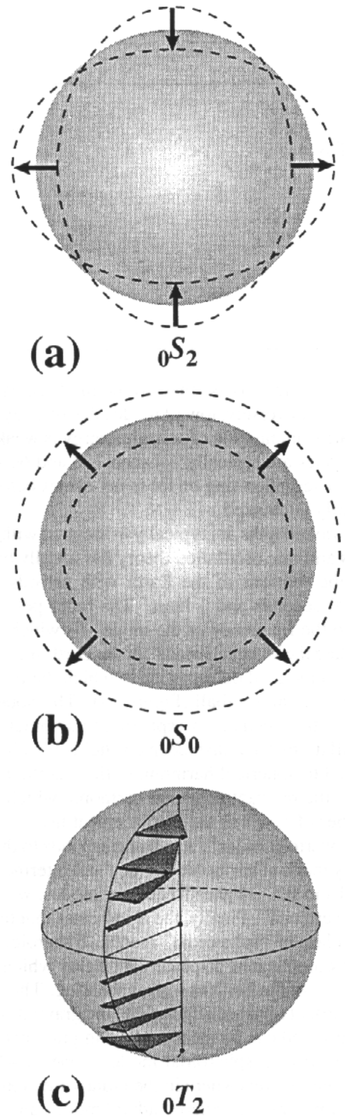


Travelling waves versus standing waves



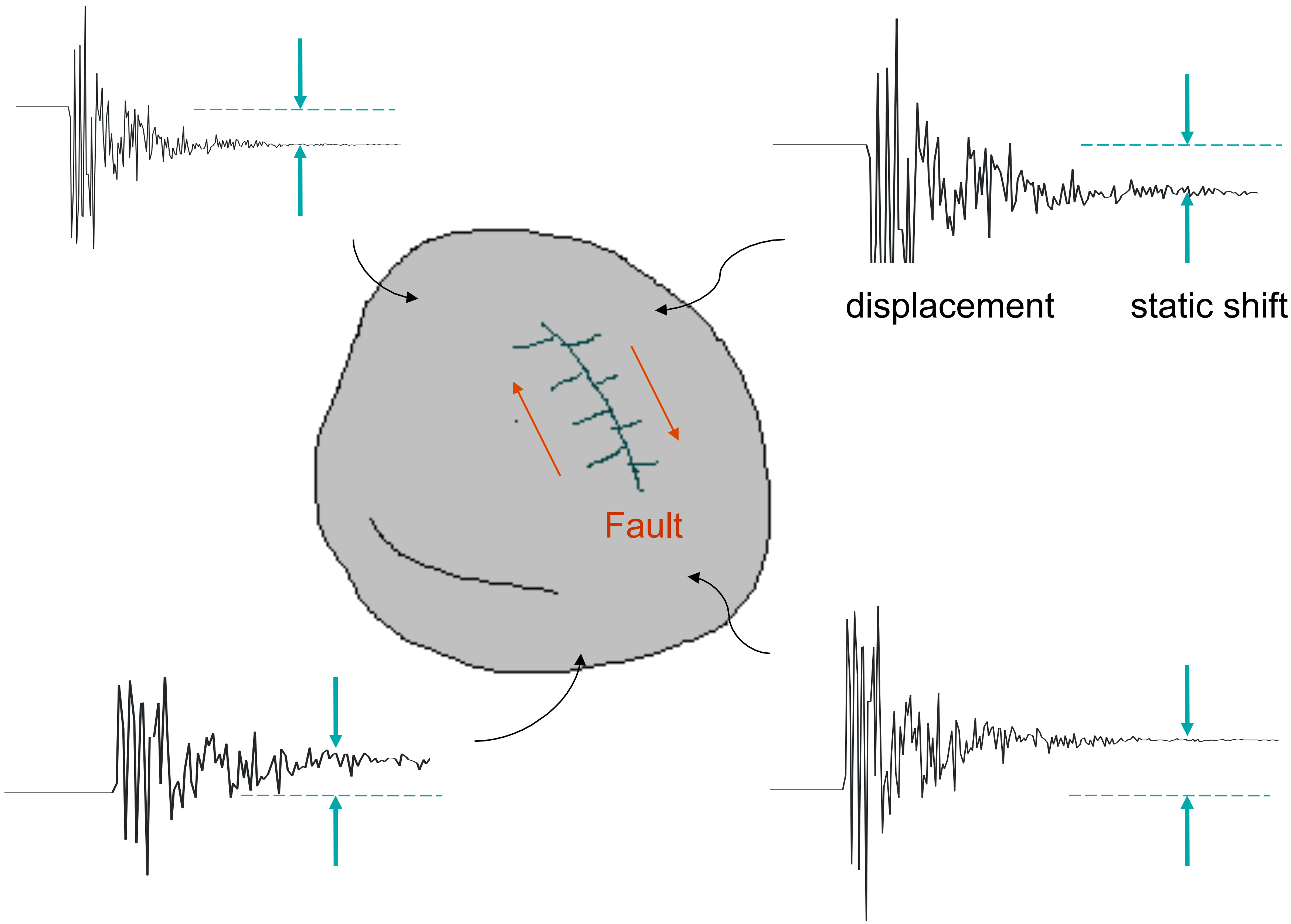
July 31, 1970 Colombian event recorded at Payson, Arizona
Linear Amplitude
Frequency [mHz]

Different classes of free oscillations



Spheroidal, Radial, and Toroidal

Static displacement produced by a “fault” in an elastic body



Earthquake Displacement Field

- Equation of motion

$$\nabla\tau + \mathbf{f_g} + \mathbf{f_s} = \rho(\mathbf{r})\,\frac{\partial^2\mathbf{u}}{\partial t^2}$$

- Solve by expanding displacement field

$$\mathbf{u}(\mathbf{r},t) = \sum_k a_k(t)\,\mathbf{u}_k^*(\mathbf{r})$$

Normal mode eigenfunctions

$$\mathbf{u}_k(\mathbf{r}) = {}_nU_l(r)\,Y_l^m(\theta,\phi)\,\hat{\mathbf{r}} + {}_nV_l(r)\,\frac{\partial Y_l^m}{\partial\theta}\,\hat{\boldsymbol{\theta}} + {}_nW_l(r)\,\frac{1}{\sin\theta}\,\frac{\partial Y_l^m}{\partial\lambda}\,\hat{\boldsymbol{\lambda}}$$

Expansion coefficients (note the static limit)

$$a_k(t) = \frac{M_o}{\omega_k^2}\,\hat{\mathbf{M}}{:}\mathrm{E}_k^*(\mathbf{r_s})[\exp(i\omega_k t) + 1]$$

Co-Seismic Displacement Field

$\mathbf{u}(\mathbf{r})$ = oscillations + static displacement

$$= 0 \text{ (as t} \rightarrow \infty) + \Sigma_{k=0}^{\infty} \omega_k^{-2} \mathbf{u}_k(\mathbf{r}) \, \mathbf{M} : \mathbf{E}_k^*(\mathbf{r}_f)$$

(Gilbert, 1970)

Eigen-mode ($\mathbf{u}_k(\mathbf{r})$, $\mathbf{E}_k(\mathbf{r}_f)$, ω_k , k = spheroidal and toroidal:

from SNREI model (e.g., 1066A, B; PREM)

Moment tensor **M**: from Global CMT catalog

The Scripps gang
Inverse theory / Normal mode

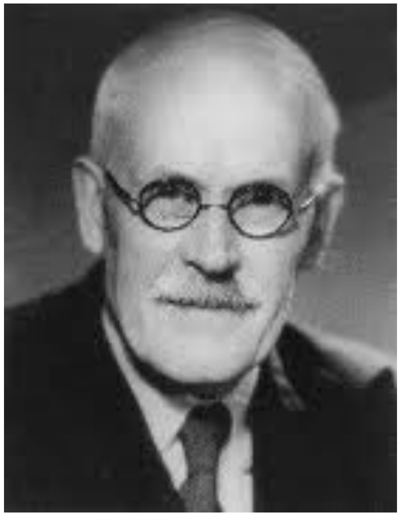

George Backus


Freeman Gilbert


Bobert Parker

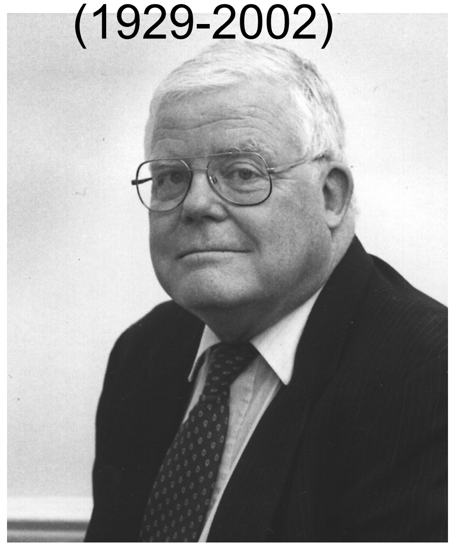

Guy Masters

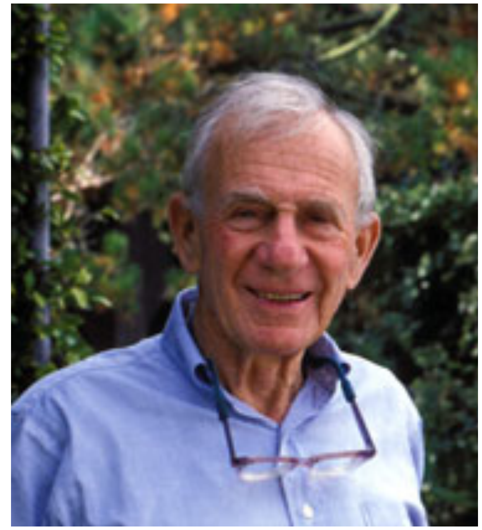
Earth Rotation



Sir Harald Jeffreys
(1891-1989)

Gordon MacDonald
(1929-2002)

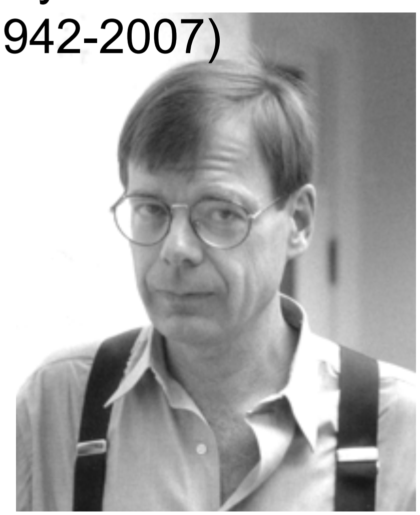




Walter Munk

Kurt Lambeck

Tony Dahlen
(1942-2007)



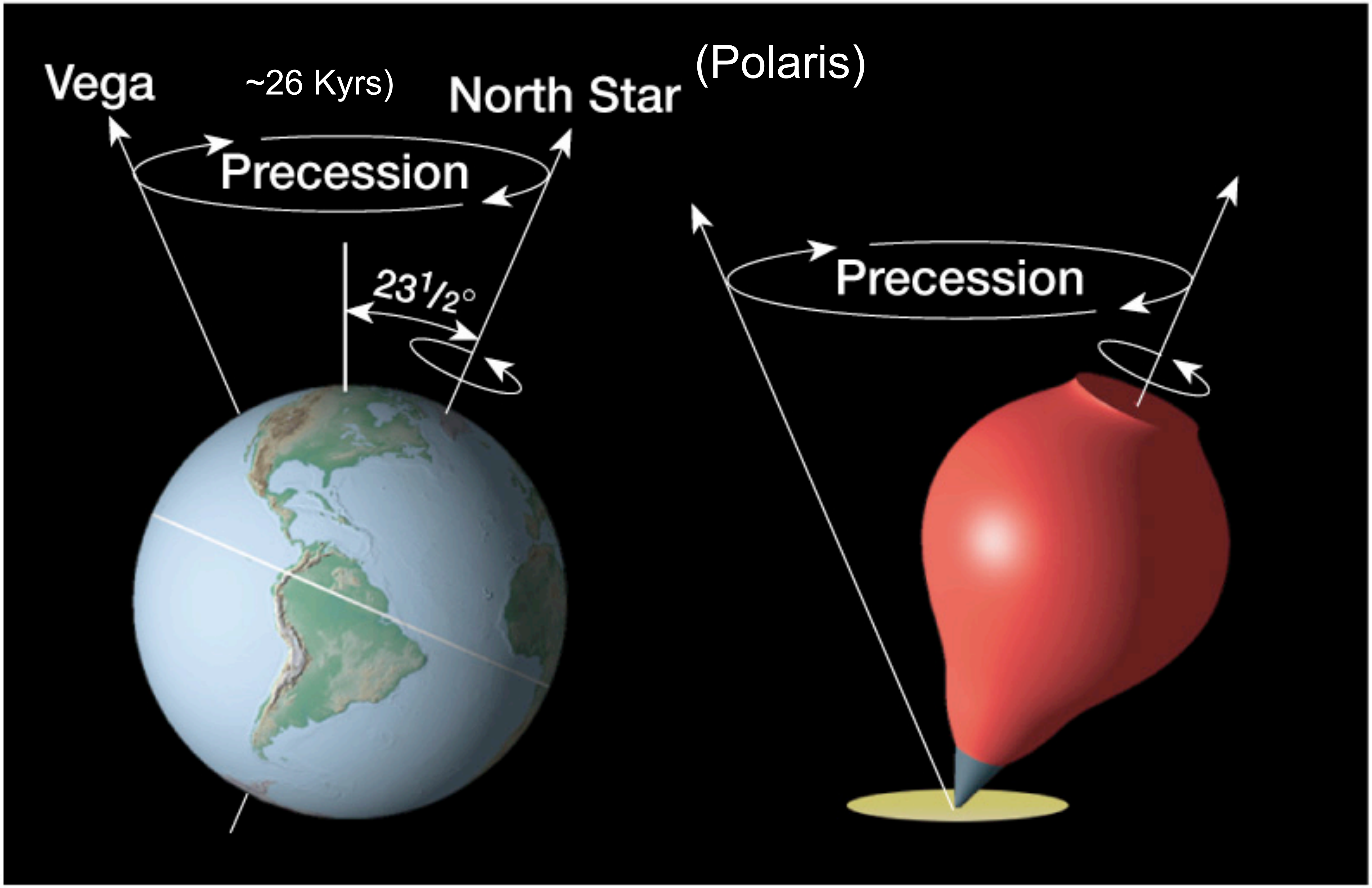
John Wahr

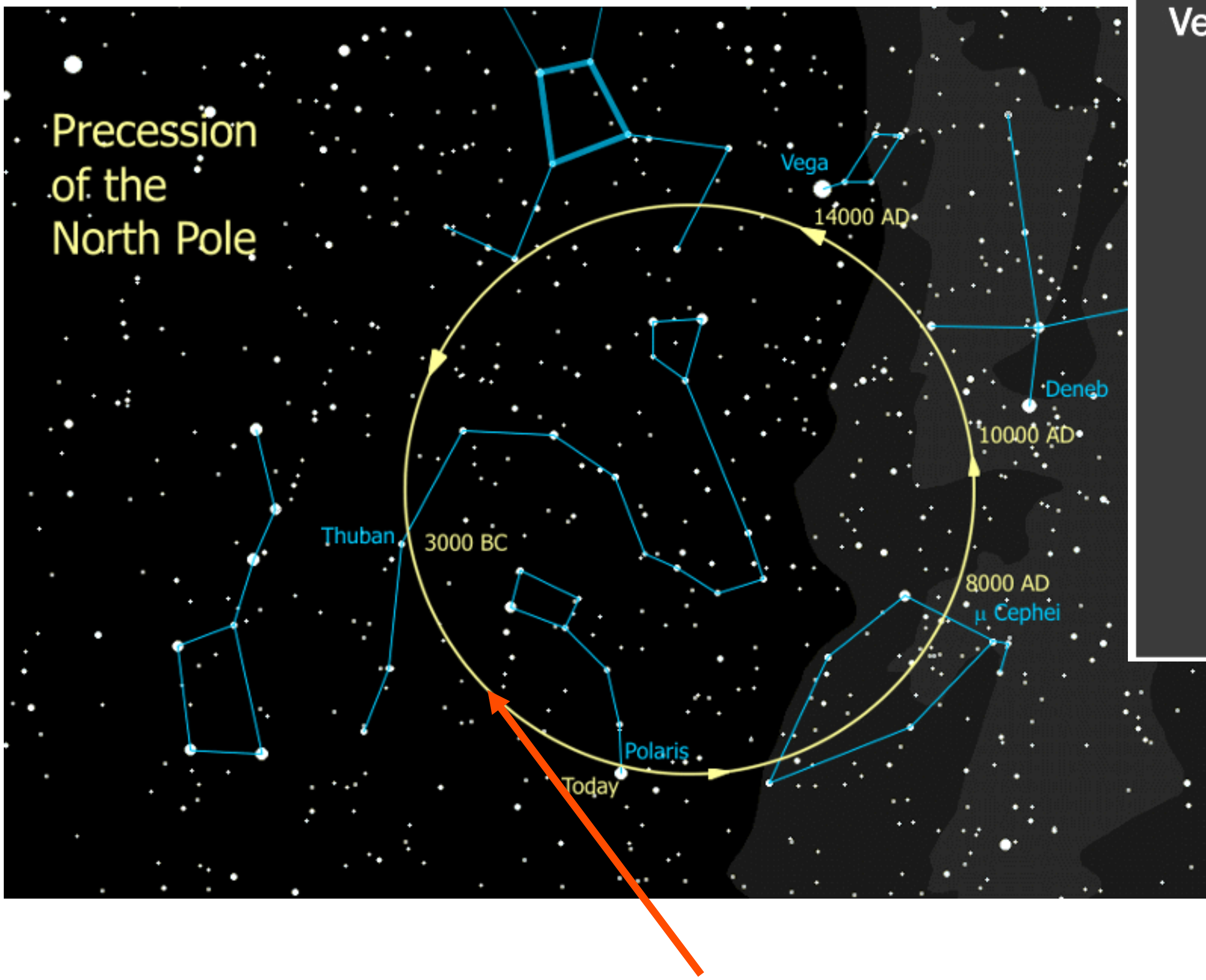
"The Earth precesses/nutates
like a top."

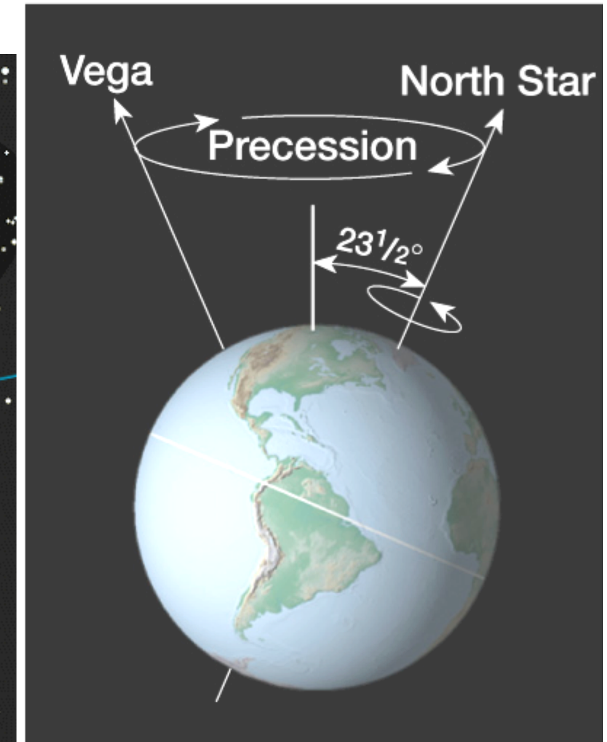
"The Earth librates
like a physical pendulum."

"The Earth wobbles
like a frisbee."

"The Earth precesses/nutates like a top."



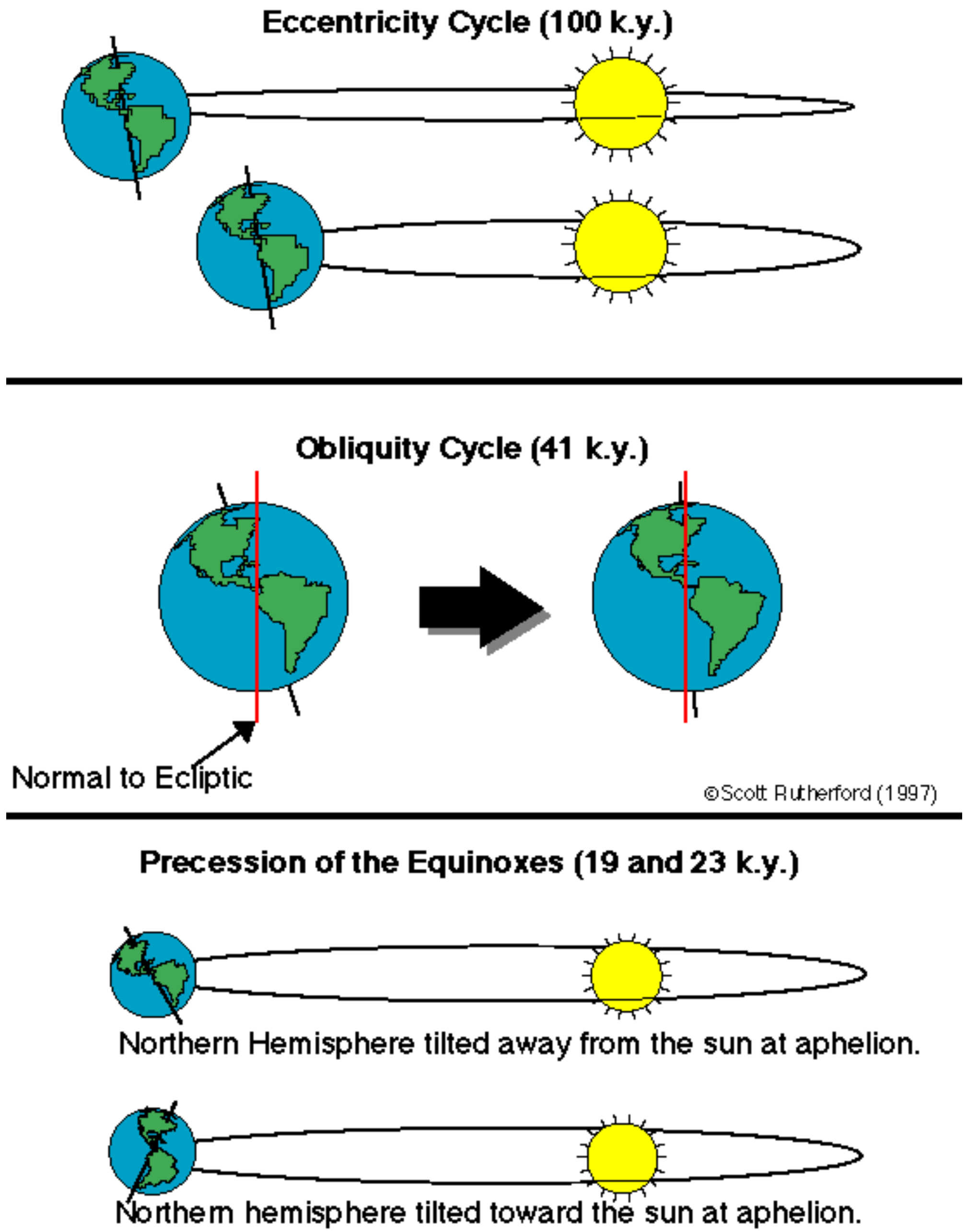




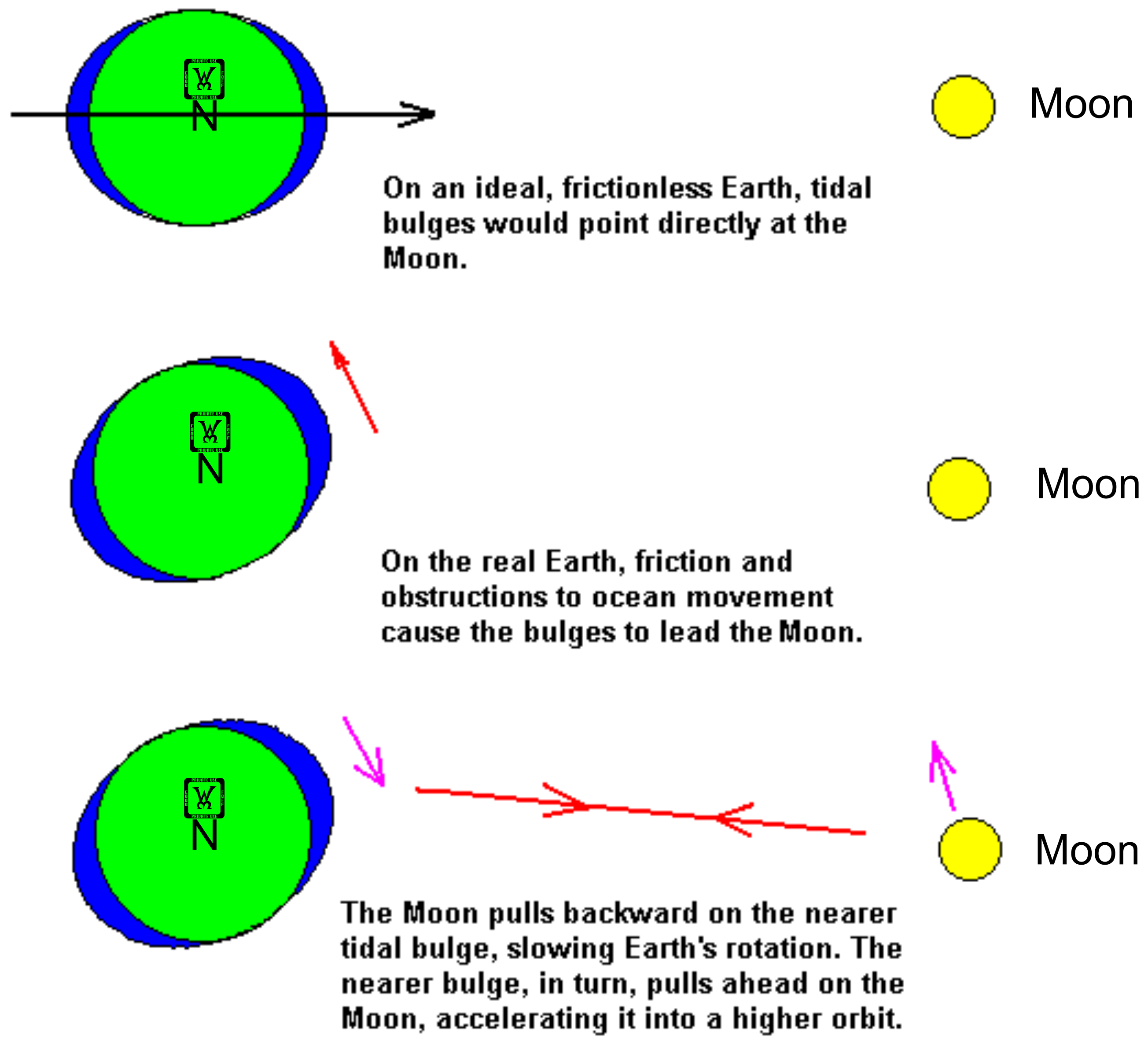
子曰：為政以德，譬如北辰，居其所，而眾星拱之。
《論語・為政》

“The Earth wobbles like a frisbee.”

Milankovitch Cycles



Tidal Braking: Slowing down Earth's rotation and pushing away the Moon



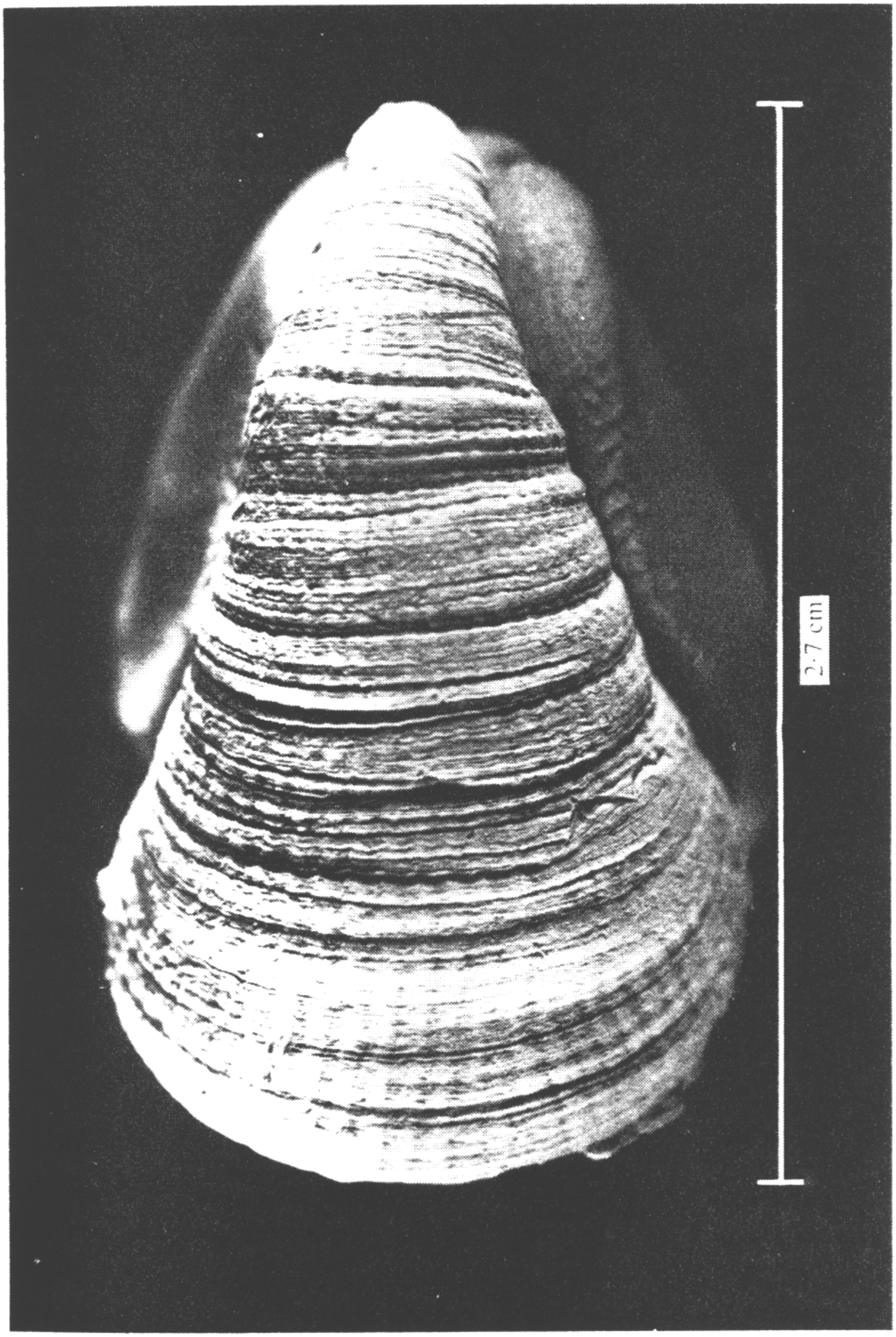
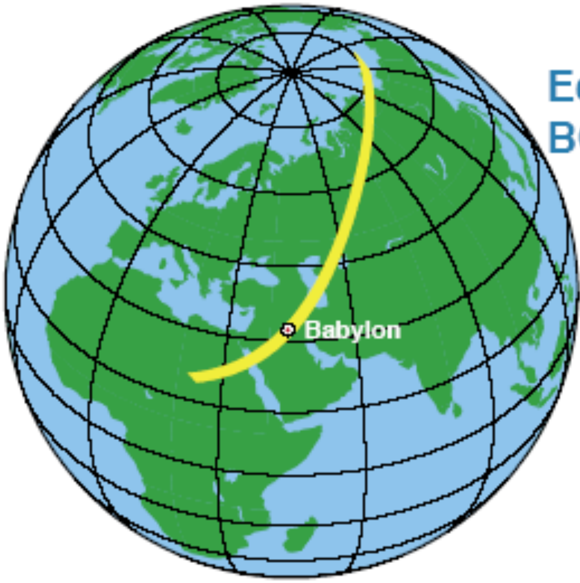


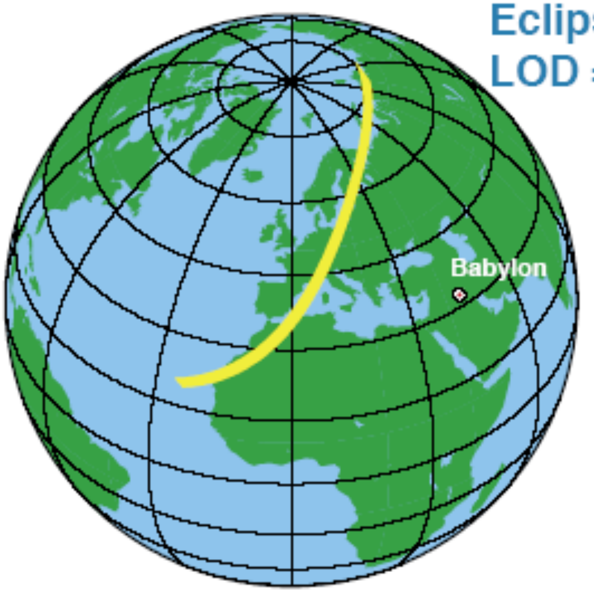
Figure 11.6. Middle Devonian coral epitheca from Michigan, U.S.A., illustrating 13 well-developed bands, each with an average of 30.8 ridges (supplied by C. T. Scrutton)

Secular Braking of Earth's Rotation

Determination of dΩ/dt from Ancient Eclipses

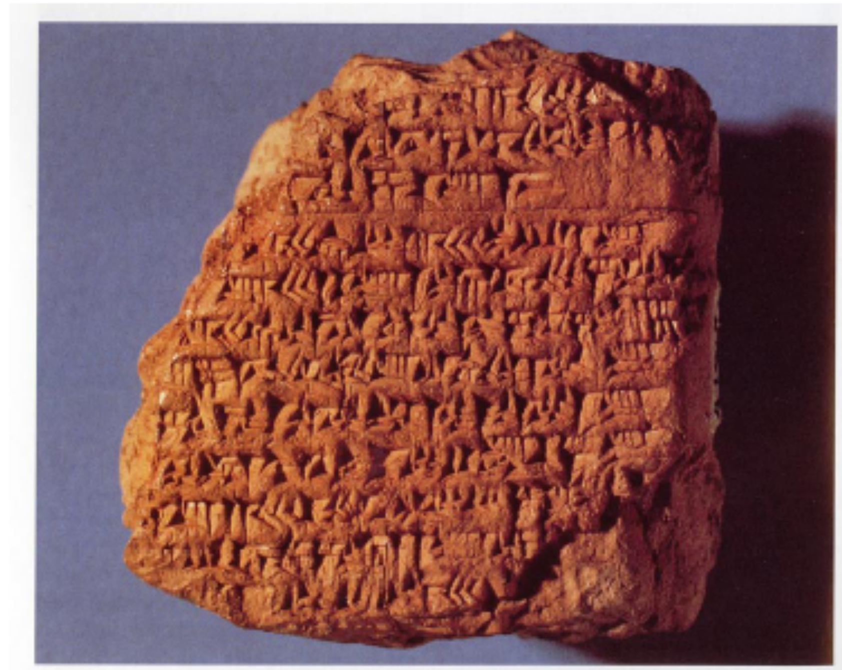


Eclipse path on
BC 136 April 15



Eclipse path if
LOD = constant

Eclipse path from Fred Espenak, GSFC



Babylonian diary from the year 87 BC (©The British Museum).

$\Delta T = 11680 \pm 460$ seconds (3.2 hrs)

(Uncertainties are strict upper/lower bounds)
(Assumes modern $dn/dt = -26''/cy^2$)

Implies $d\Lambda/dt = 1.71 \pm 0.07$ ms/century

A Babylonian day was ~37 ms shorter than ours.

A very precise estimate from one single eclipse!

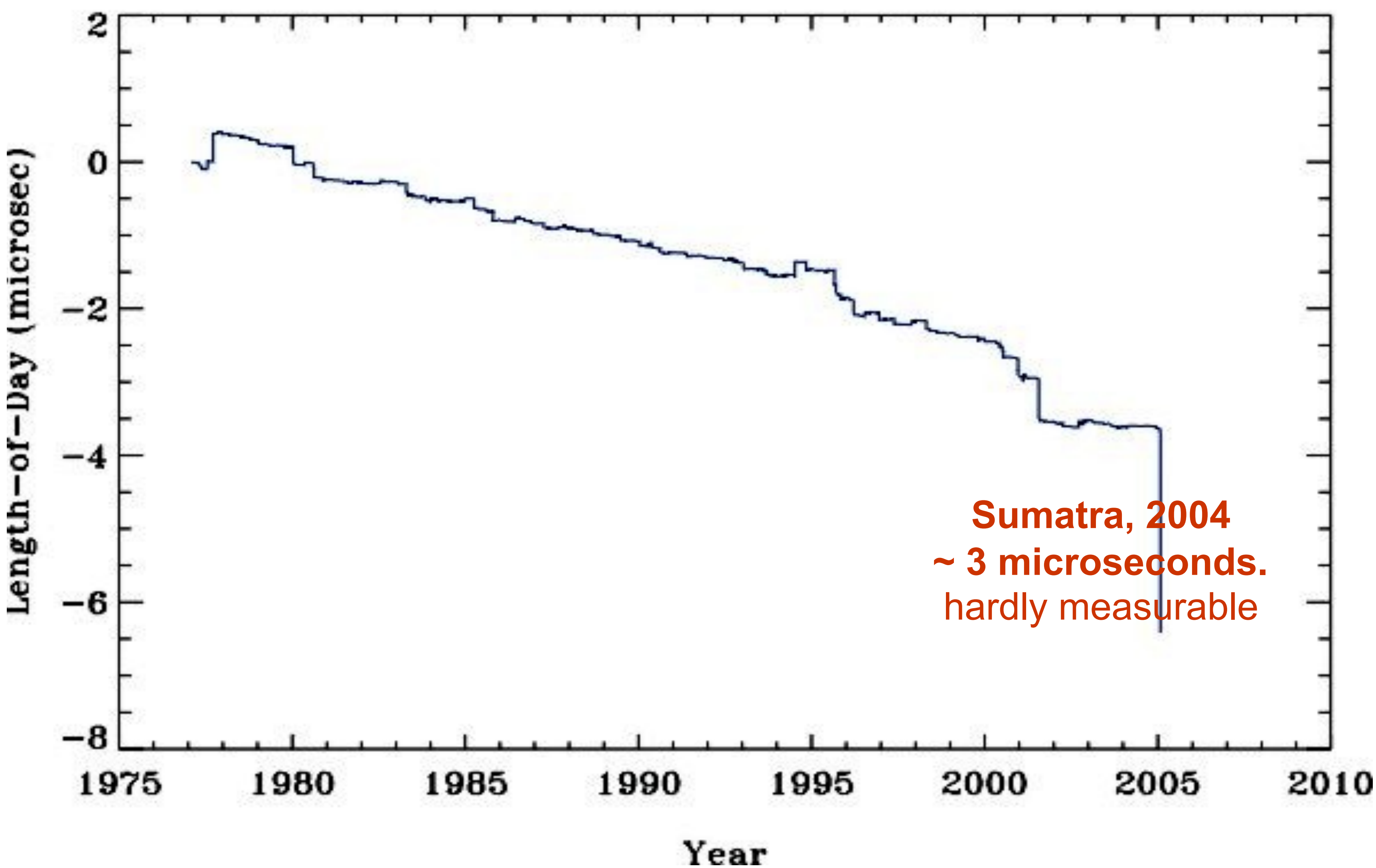
Co-Seismic Effects on Earth's Rotation

- Milne (1906); Cecchini (1928)
- Munk & MacDonald (1960)
- Alaska earthquake (1964), Press (1965)
- **Mansinha & Smylie (1967; +)**
- **Ben-Menahem & Israel (1970; +)**
- **Rice & Chinnery (1972)**
- **Dahlen (1973)**
- **Dziewonski & O'Connell (1975)**
- **Smith (1977)**
- **Souriau & Cazenave (1985)**
- **Gross (1986)**
- **Chao & Gross (1987; +)**
  - using normal mode summation (Gilbert, 1970)
  - In terms of seismic moment tensor (Harvard CMT catalog)
  - need (SNREI) Earth model (PREM) and normal mode eigen-functions (Masters)
  - similar formulas for changes in gravity field, energy, etc.

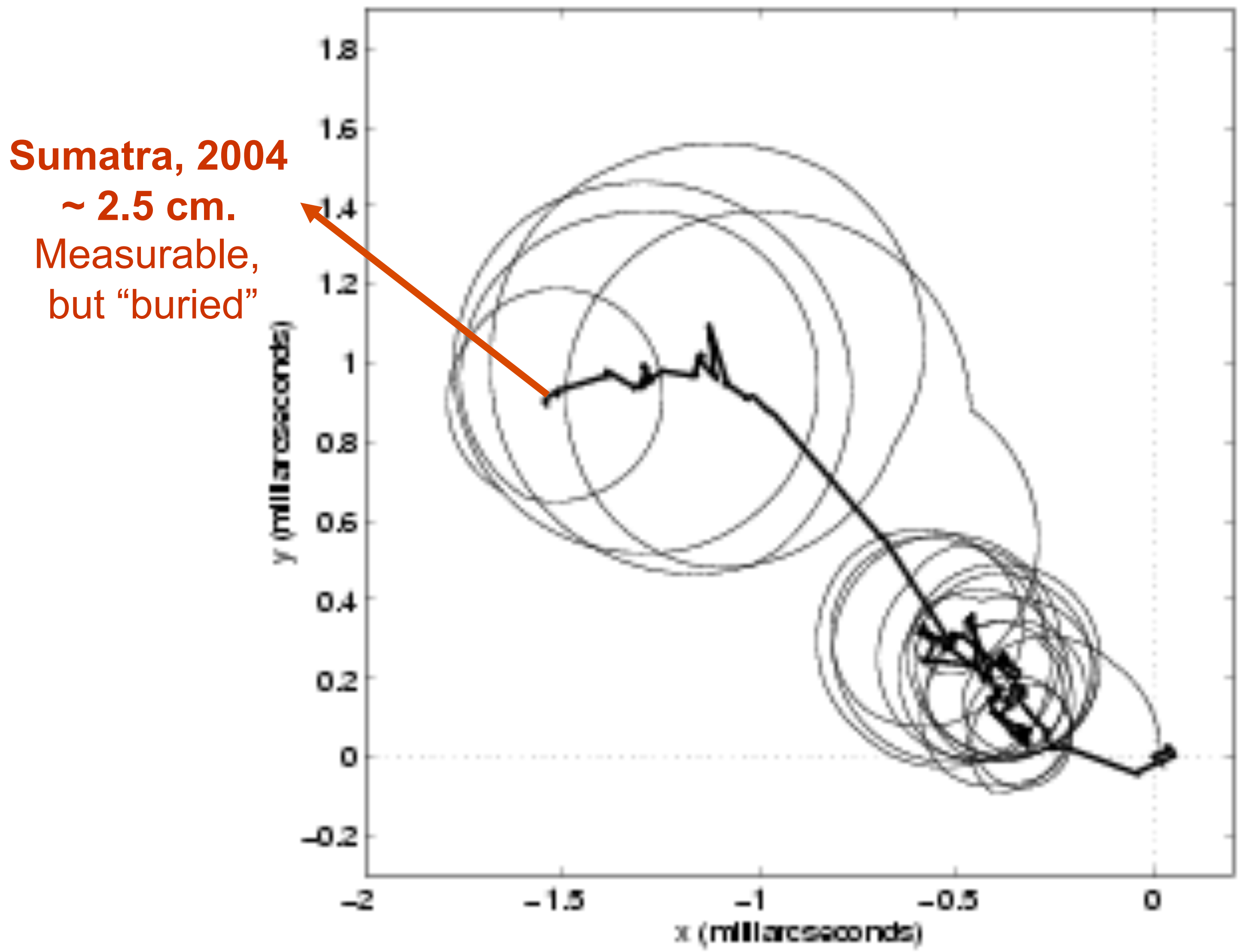
Seismic Moment Tensor

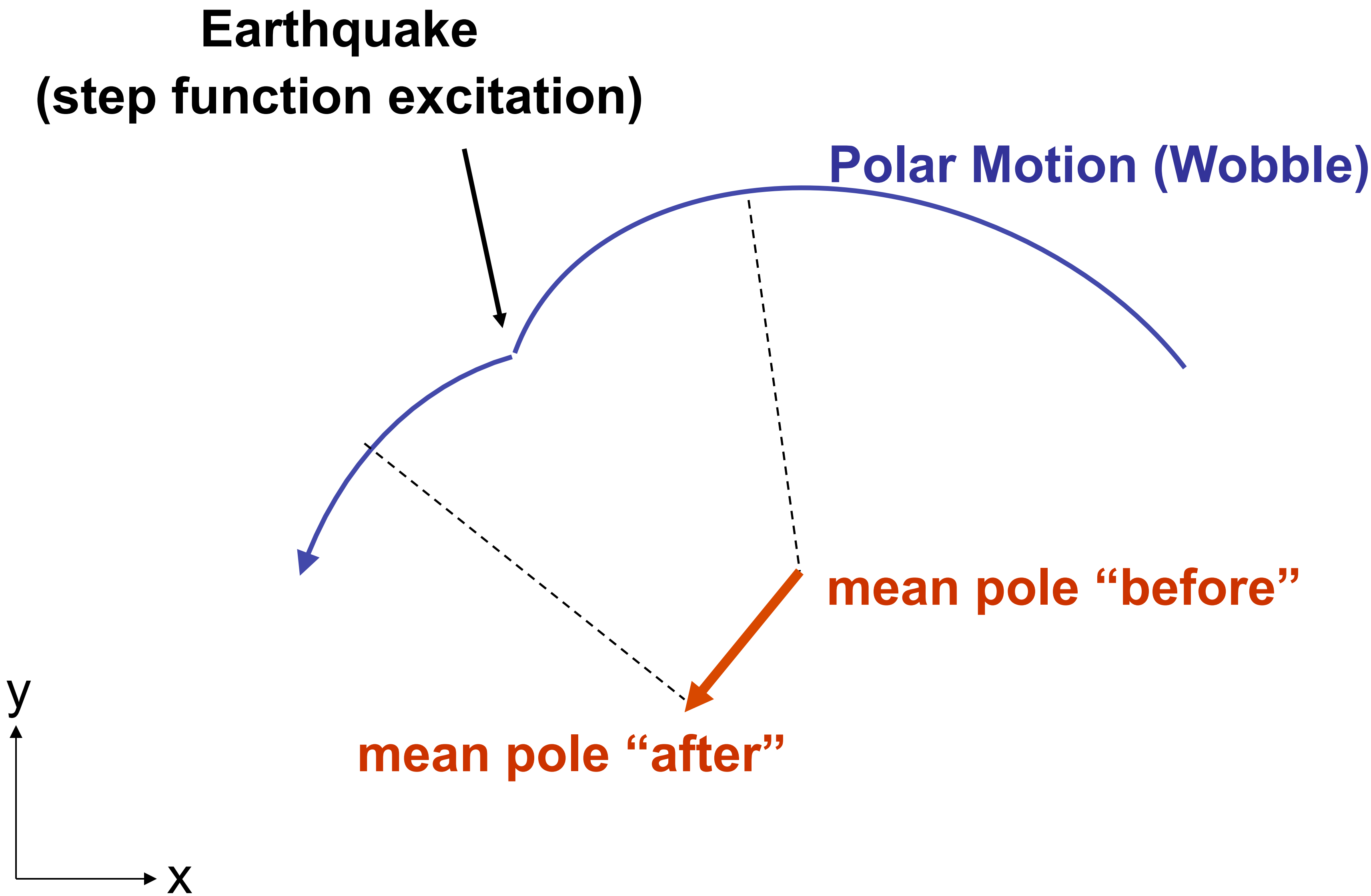
- Contains all information about source mechanism (magnitude, fault direction, slip angles, etc.)
- 2nd-order tensor (conservation of linear momentum)
- Symmetric tensor, only 6 independent parameters (conservation of angular momentum)
- Magnitude (seismic moment) $[M:M]^{1/2}$ is a good measure of earthquake size
  => moment scale (vs. Richter scale)

Cumulative change in Length-of-Day by earthquakes since 1976

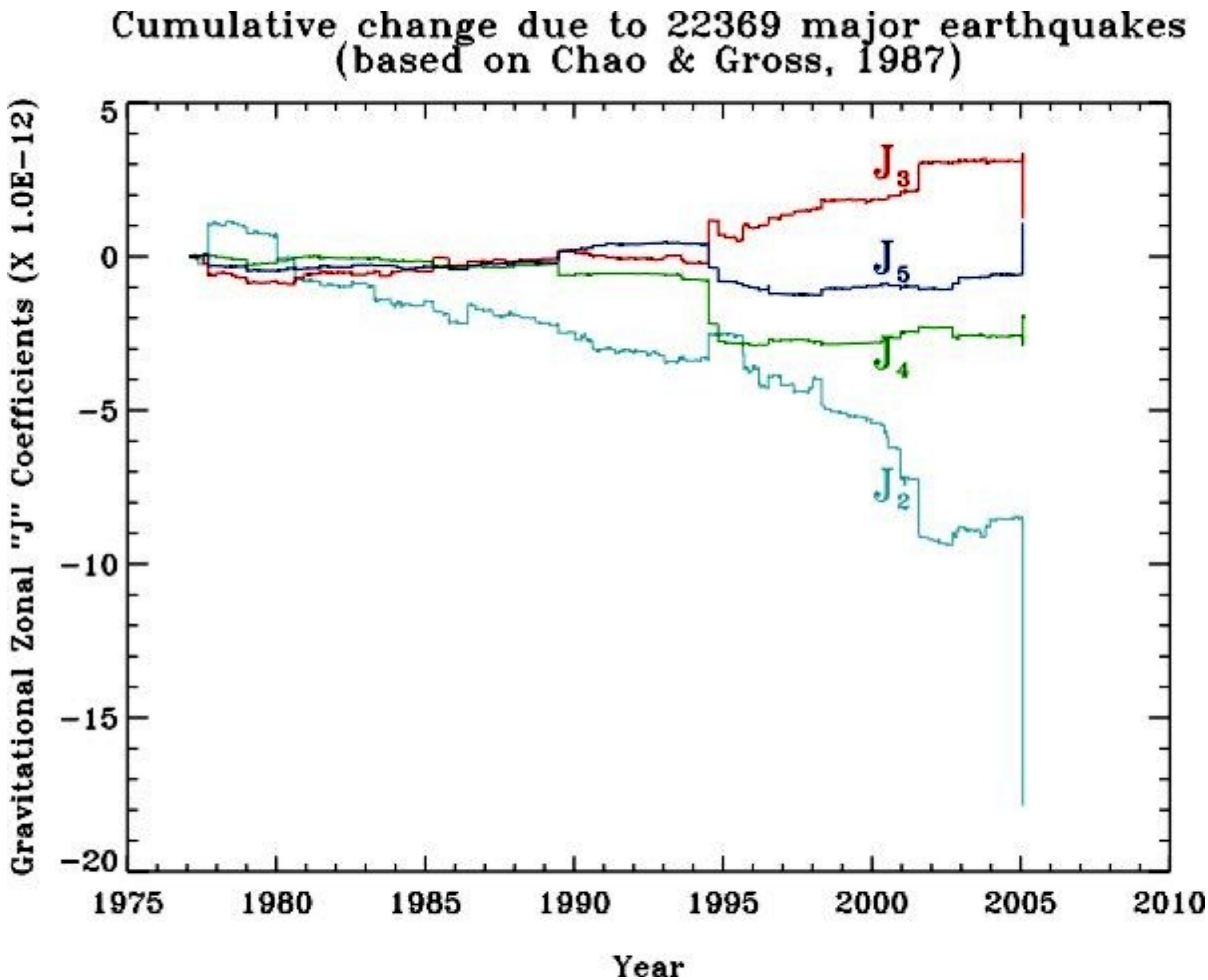


Cumulative "Mean" Pole Position shift by earthquakes, 1976-1999

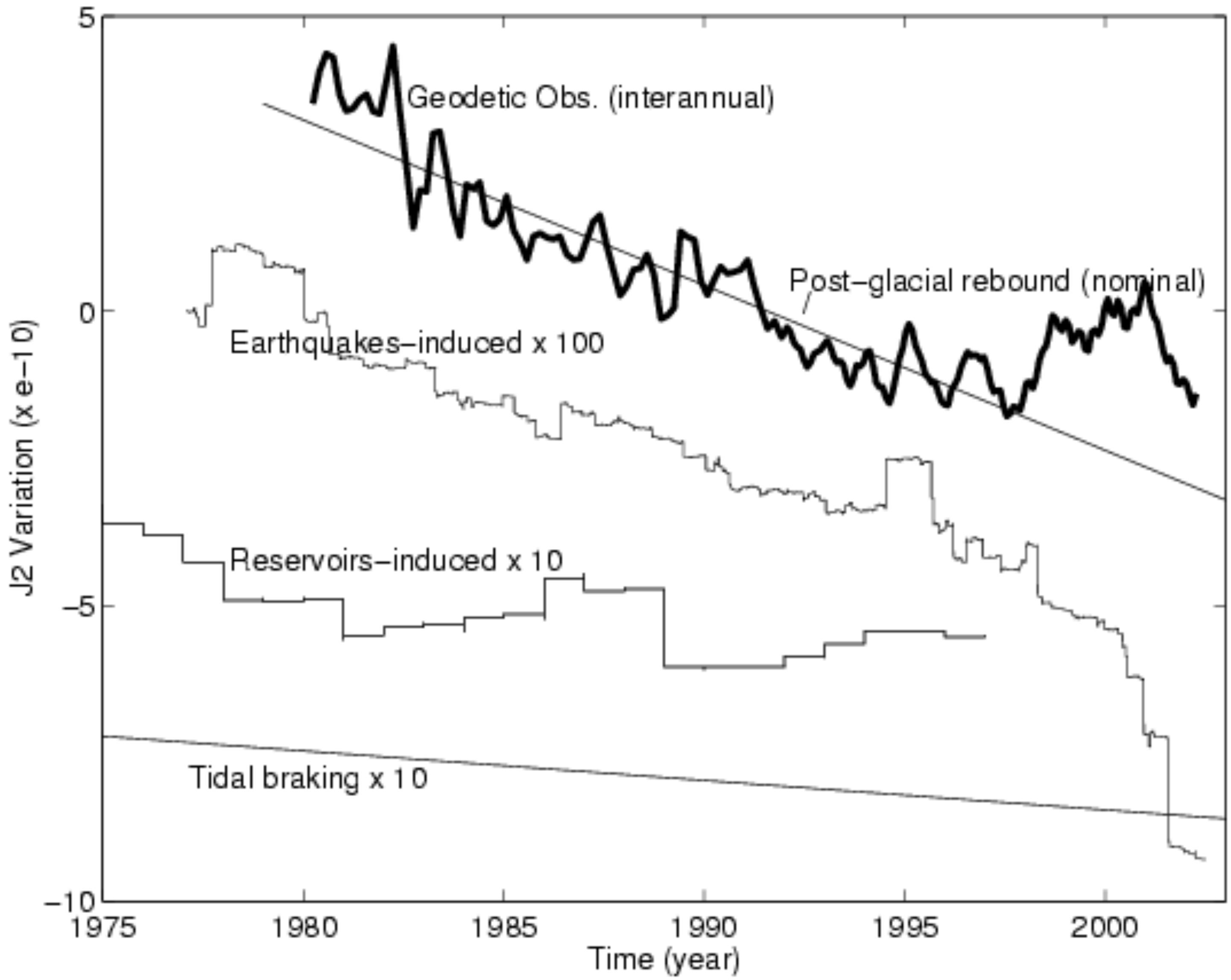



Earthquake
(step function excitation)
Polar Motion (Wobble)
mean pole “before”
mean pole “after”
y
x

發生時間	地點	地震規模	日長改變量	自轉軸偏移量	自轉軸偏移方向
1957/3/9	阿拉斯加	9.1	(資料不足)	(資料不足)	(資料不足)
1960/5/22	智利	9.5	–8.4 μs	68 cm	115 °E
1964/3/27	阿拉斯加	9.2	+ 6.8 μs	23 cm	198 °E
2004/12/26	蘇門答臘	9.3	–6.8 μs	7 cm	127 °E
2010/2/27	智利	8.8	–1.3 μs	8 cm	112 °E
2011/3/11	日本	9.1	–1.6 μs	15 cm	139 °E


Cumulative change due to 22369 major earthquakes
(based on Chao & Gross, 1987)
Gravitational Zonal "J" Coefficients (X 1.0E−12)
Year
5
0
−5
−10
−15
−20
1975
1980
1985
1990
1995
2000
2005
2010
J_3
J_5
J_4
J_2

Long-term changes in Earth's oblateness J_2



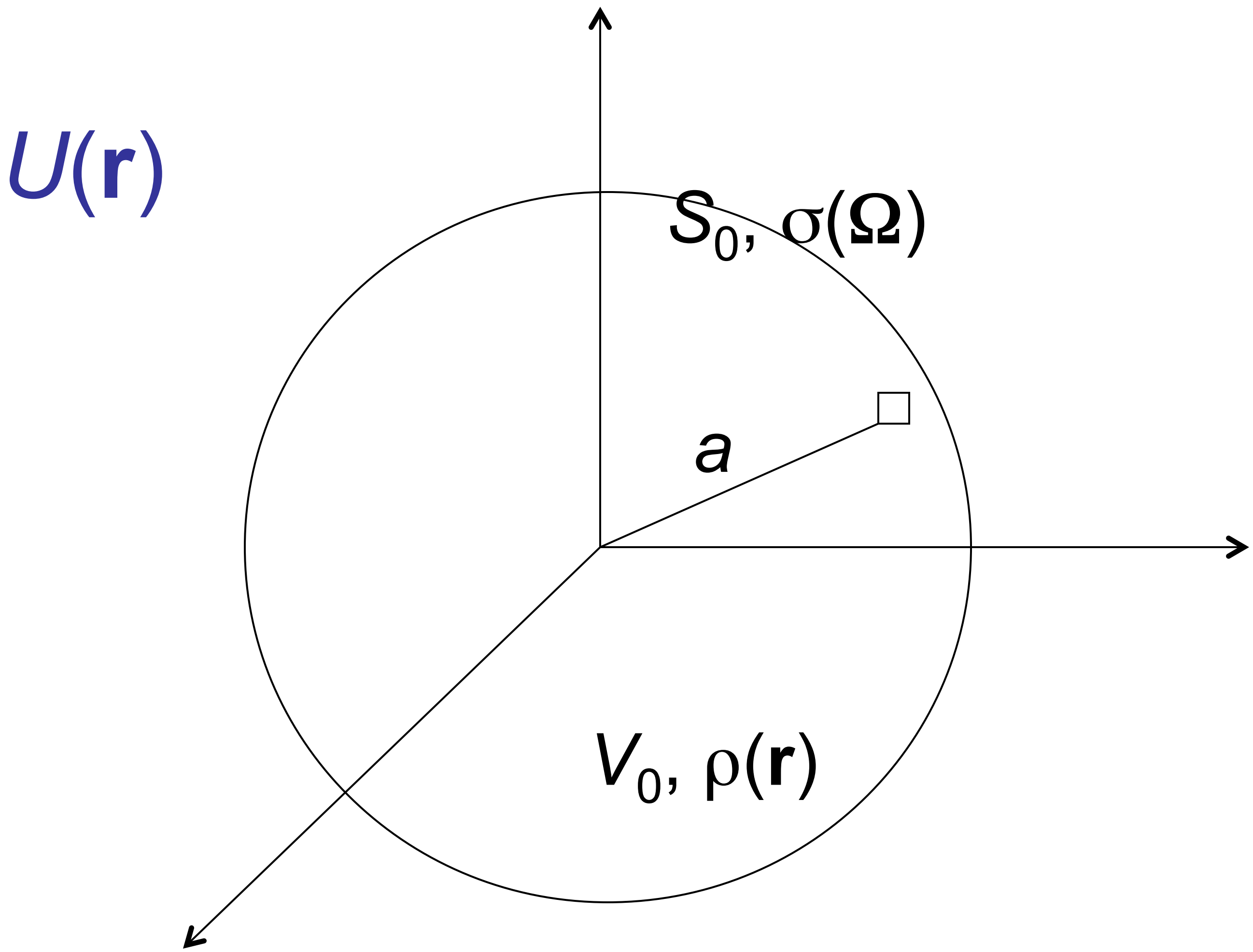
Limitation of (time-variable) gravity signal

"You don't know where it comes from!"

- Low spatial resolution

- Non-uniqueness in inversion

- Sum of all sources



U(r)
S0, σ(Ω)
a
V0, ρ(r)

Gravitational Potential Field

- Newton's gravitational law

$$U(\mathbf{r}) = G \iiint_{V_0} \frac{\rho(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|} dV_0$$

- Addition theorem

$$1/|\mathbf{r} - \mathbf{r}_0| = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1)^{-1} (r_0^{\,n} / r^{n+1}) Y_{nm}^{\;*}(\Omega) Y_{nm}(\Omega_0)$$

⇨ Multipole expansion of gravity field

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)\, r^{n+1}} \left[\iiint_{V_0} \rho(\mathbf{r}_0)\, r_0^{\,n}\, Y_{nm}(\Omega_0)\, dV_0 \right] Y_{nm}^{\;*}(\Omega)$$

Gravitational Potential Field (Geoid)

- Multipole expansion of Newton's formula:

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)\, r^{n+1}} \left[\iiint_{V_0} \rho(\mathbf{r}_0)\, r_0^{\,n}\, Y_{nm}(\Omega_0)\, dV_0 \right] Y_{nm}^{\;*}(\Omega)$$

- Conventional expression (satisfying Laplace Eq. in terms of Stokes Coeff.):

$$U(r,\theta,\lambda) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_{nm}(\cos\theta) \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right)$$

$$\Rightarrow \quad \boxed{C_{nm} + iS_{nm} = \frac{1}{(2n+1)\, M a^n} \iiint_{V_0} \rho(\mathbf{r})\, r^n\, Y_{nm}(\Omega)\, dV}$$

3-D Gravitational Inversion

- Multipole expansion

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)\, r^{n+1}} \left[\iiint_{V_0} \rho(\mathbf{r}_0)\, r_0^{\,n}\, Y_{nm}(\Omega_0)\, dV_0 \right] Y_{nm}^{\;*}(\Omega)$$

2*n* +1 (known) multipoles for each degree *n*

- Moment expansion

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{\alpha,\beta,\gamma\ge 0}^{\alpha+\beta+\gamma=n} \frac{(-1)^n}{\alpha!\,\beta!\,\gamma!} \left[\iiint_{V_0} x_0^{\alpha}\, y_0^{\beta}\, z_0^{\gamma}\, \rho(\mathbf{r}_0)\, dV_0 \right] \frac{\partial^n}{\partial x^{\alpha} \partial y^{\beta} \partial z^{\gamma}} \left(\frac{1}{|\mathbf{r}|} \right)$$

(*n*+1)(*n*+2)/2 (unknown) moments for each *n*

⇨ "Degree of deficiency" of knowledge is *n*(*n*-1)/2 for each degree *n*.

Degree n	# multipoles $(2n + 1)$	# moments $(n+1)(n+2)/2$	Degree of deficiency $n(n-1)/2$
0	1 (monopole)	1 (total mass)	0
1	3 (dipole)	3 (center of mass)	0
2	5 (quadrupole)	6 (inertia tensor)	1
3	7 (octupole)	10 (3rd moment)	3
4	9	15	6
5	11	21	10
6	13	28	15
100	201	5151	4950

The degree of deficiency as a function of spherical harmonic degree n in the 3-D gravitational inversion.

Additional physical/mathematical constraints leading to unique solutions:

- minimum shear energy
- maximum entropy of ρ
- minimum norm-2 variance for the lateral distribution
- ………..

2-D gravitational Inversion on a spherical shell S_0

$$C_{nm} + iS_{nm} = \frac{a^2}{(2n+1)M} \iint_{S_0} \sigma(\Omega)\, Y_{nm}(\Omega)\, d\Omega$$

$$\sigma(\Omega) = \Sigma_{n,m} \sigma_{nm} Y^*_{nm}(\Omega)$$

$$\Rightarrow \quad \sigma_{nm} = \frac{(2n+1)M}{4\pi a^2}\left(C_{nm} + iS_{nm}\right)$$

It is possible to mimic ANY external field by means of some proper surface density on S_0.

2-D gravitational Inversion on a spherical shell S_0 (cont'd)

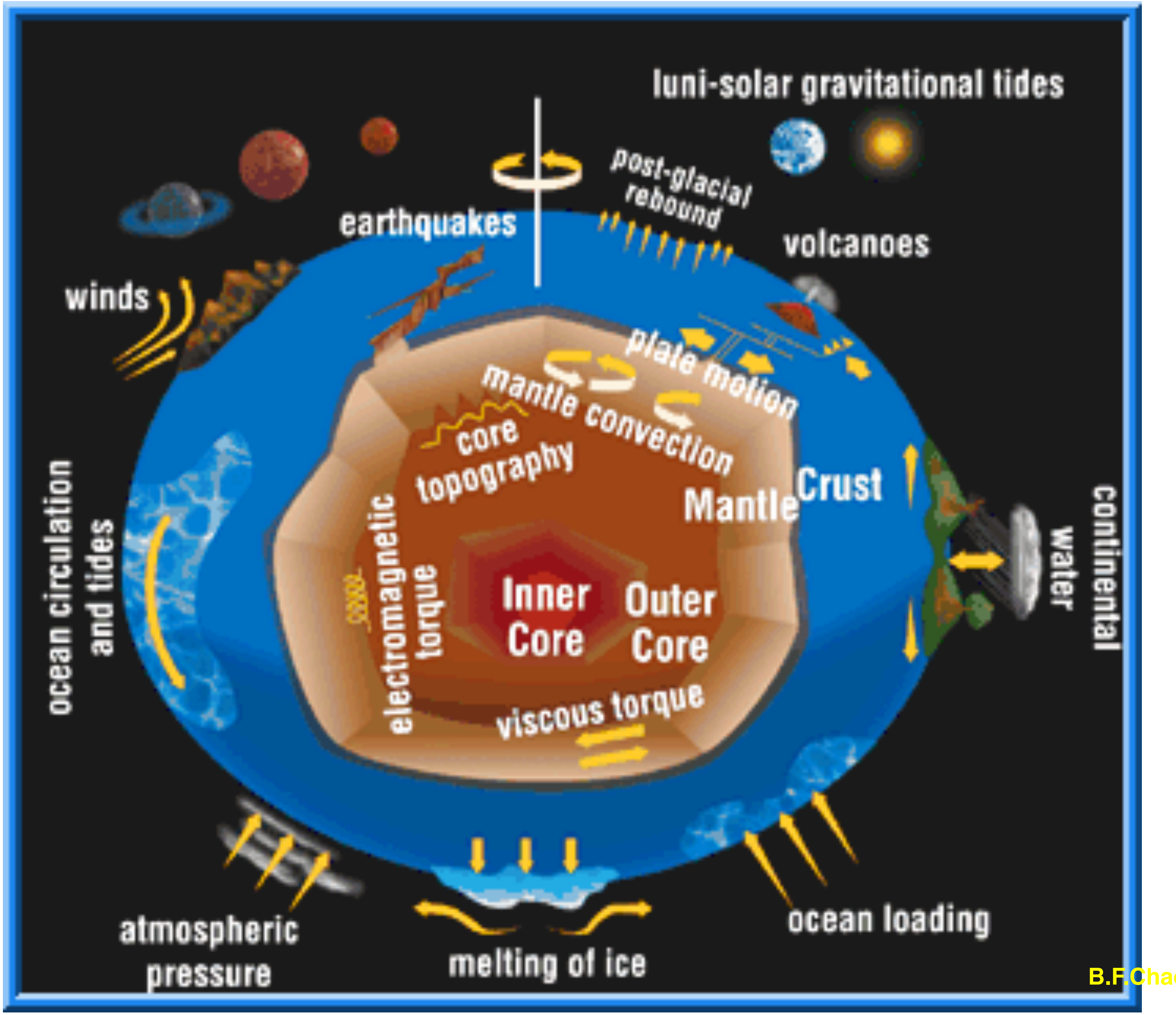
For CHANGES due to mass redistribution on S_0 (taking into account of loading effect), in Eulerian description:

$$\Delta C_{nm}(t) + i\,\Delta S_{nm}(t) = \frac{a^2(1+k'_n)}{(2n+1)M} \iint \Delta\sigma(\Omega;t)\, Y_{nm}(\Omega)\, d\Omega$$

$$\Rightarrow \boxed{\Delta\sigma_{nm}(t) = \frac{(2n+1)M}{4\pi a^2(1+k'_n)}\left[\Delta C_{nm}(t) + i\,\Delta S_{nm}(t)\right]}$$

Unique!

Loading effect "undone"



luni-solar gravitational tides
post-glacial rebound
earthquakes
volcanoes
winds
plate motion
mantle convection
core topography
Crust
Mantle
electromagnetic torque
Inner Core
Outer Core
viscous torque
ocean circulation and tides
continental water
atmospheric pressure
melting of ice
ocean loading
B.F.Chao

(near) Surface Mass Transports

- Earth ellipticity ~ ½ of 1/300 a ~ 10 km
- Atmosphere scale height ~ 10 km
- Ocean < ~ 5 km
- Land hydrology < a few km
- Crustal/topography change < ~ 30 km

Nice things about spherical harmonics:

- Wavelength, or spatial resolution,

  ~ 40,000/2N km → Concept of spectrum
- Altitude attenuation ~ r^{-n+1}
- Geoid: Stokes Coeff. (C_{nm}, S_{nm})
- Gravity Disturbance: $(n+1)*(C_{nm}, S_{nm})$
- Gravity Anomaly: $(n-1)*(C_{nm}, S_{nm})$
- Surface Mass Change: $(2n+1)*(\Delta C_{nm}, \Delta S_{nm})$

Conclusions

for the [external gravity => mass density] inversion:

- The 3-D inversion is non-unique (well-known).
- This 3-D non-uniqueness is associated with the radial (depth) dimension.
- Comparing the (spherical harmonic) multipole expansion and the moment expansion => The degree of deficiency in inversion is $n(n-1)/2$ for each degree n.
- The 2-D inversion on a spherical shell is unique.
- In terms of spherical harmonics this 2-D uniqueness is convenient and useful in (global) time-variable gravity studies (such as GRACE).